Become O'Sullivan Mathematically: Model the motion of cue ball

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Abstract

Billiards, or pool is a time honored sport that has great popularity among the world. Behind the interesting games, one may observe its profound physics and mathematics nature. In particular, the motion of the cue ball after being hit by the cue stick is a great scenario to model. This project solves this process in a $1d$ sense. In this project, we first solve the velocity and angular velocity of the cue ball analytically. With the analytical results, we then give estimations of some crucial constant parameters(such as friction coefficient.) Also, a numerical stimulation of the process is designed based on the outcome of the effort above. Last but not least, we also design a program that generates a video clip of the motion of cue ball given any initial condition as input.

Introduction

Billiards, or pool, is a time-honored sport that has myriad branches including Snooker, 9-balls, and Chinese 8-Balls, etc. Despite different rules of different games, the main mechanism is quite clear: we are given a cue stick with which we need to cue (or hit) the cue ball to let it hit the target ball into the pocket. However, making the target ball fall into the pocket is not all of the game. Sometimes, we need to let the cue ball go to some certain position after hitting the target ball, which will benefit the next cueing. For example, professional players like Ronnie O'Sullivan sometimes make the cue ball goes back to its initial position after hitting the target ball in the long distance. The terminology for this skill is back-spinning.

But what's the cause of back-spinning? Generally, it is because the ball is spinning backwardly before it hits the target ball. In this project, we are going to solve the motion of the cue ball after hitted by the cue stick, which means what's the velocity of the cue ball and how does it rotate(spin) at any time before it stops. Though we don't take the ball-ball consideration and ball-cusion interaction in this project, the result of this project will help us to do this two topics (as we will know the motion of the cue ball at any time).

In this project, we are going to investigate the 1 dimensional case of the version.

Methodology

As is mentioned in the introduction part, we only model the $1-dim$ case **i.e.** we only consider the case that the stick(parallel to the ground) hit on the middle line (axis of symmetry) of the cue ball(thus generating the velocity and angular velocity of $1d$).

The essential question of our model, therefore, is given the impulse provided by the cue stick (denoted $J(kg \cdot m/s)$) and the position (here we use its vertical distance to the center of the cue ball, denoted h), we want to know the motion (velocity and angular velocity) of the cue ball at any time before it stops. Here's the visualization of our model (see Fig.1).

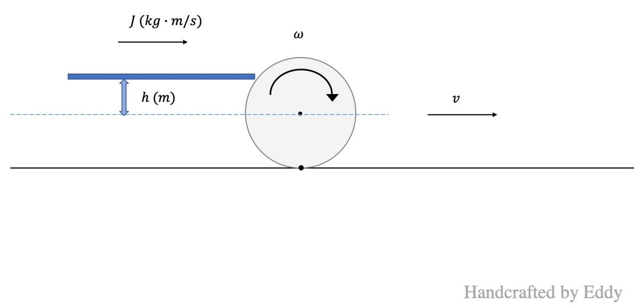

Figure 1: Model

This task can be divided into two sub-tasks, which are independent to some extent.

- The cueing process. After cueing, what kind of initial condition does the cue ball have? (or how does J and h have effect on initial velocity v_0 and initial angular velocity ω_0)
- Given the initial condition of cue ball, what velocity v and angular velocity ω does the cue ball have at any time before it stops. (With the result we also have access to the displacement and phase of the cue ball.)

The variable and parameters we will is listed in the table below.

Parameters	Unit	Range	Comment
m	kg	constant	The mass of the cue ball
r	m	constant	The radius of the cue ball
J	$kg \cdot m/s$	≥ 0	The impulse provided by the cue stick
h	m	(-r, r)	Hitting position of the cue stick, positive means above the center of the mass
v	m/s	≥ 0	The velocity of the cue ball, a function of time t, positive means the direction is toward right
ω	rad/s	(-, +)	The angular velocity of the cue ball, a function of time t, positive means **anti-clockwise**
μ	dimensionless	(0, 1)	(Sliding) friction coefficient

The main contributions of this project are

1. Solved the velocity and angular velocity of the cue ball before it stops analytically (given initial condition).
2. Estimates the actual constant parameters (**e.g. friction coefficient**) during an actual snooker game
3. Simulate the cueing process (until the cue ball stops) numerically.
4. (The most interesting part) We designed a program to generate the video clip of cueing. That is to say, input any impulse J and the hitting position h, a video clip depicting how the ball slides and rolls will be generated automatically.

Now let's come to the first contribution mentioned.

Motion of the cue ball: Given initial condition

In this part, we are to solve the cue ball's motion given the initial velocity v_0 and the initial angular velocity ω_0.

Force analysis

We first do the force analysis of the model.(see Fig.2)

As we can see, friction is the only external force (having effect on v and ω).

The magnitude is given by the formula $f = \mu mg$, while its direction is nontrivial. We do force analysis on the contact point between cue ball and table, denoted A. The velocity of A with respect to the table consists of 2 velocity:

- velocity of the ball as a point of mass, which is v
- velocity caused by spinning, which is $\omega \cdot r$

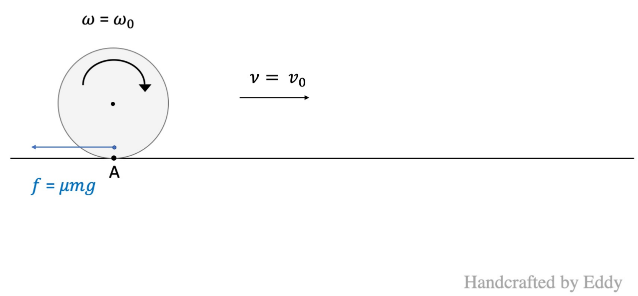

Figure 2: Force analysis

In that case, the velocity of A against table is $v + \omega \cdot r$. As the direction of friction is against the velocity, the direction of f can be expressed as a function of v and w (sgn function returns ± 1, for positive and negative respectively, returns 0 for 0):

$$sgn_f(v, \omega) = -sgn(v + w \cdot r)$$

Given this condition, we are able to say the velocity and angular velocity is governed by

$$\frac{dv}{dt} = sgn_f(v, \omega) \cdot \mu g$$

$$\frac{d\omega}{dt} = sgn_f(v, \omega) \cdot \frac{\mu mgr}{I} = sgn_f(v, \omega) \cdot \frac{\mu mgr}{\frac{2}{5}mr^2} = sgn_f(v, w) \cdot \frac{5\mu g}{2r}$$

(Note that here we use the fomula $dw/dt = torque/inertia$, where the inertia for a solid ball is $2/5mr^2$)

One may found it is too complicated to write it into the differential equation. In fact, they are merely ODE with constant growth rate. However, by writing in this form, we emphasize the v and w are interrelated(by the term $sgn_f(w, v)$). Also, by writing it into ODE forms, one can approximate it numerically by the package `scipy.integrate.odeint`. What's more, if we endowed some ODE view on this matter, we can see the state $v + \omega \cdot r$ is in some way a fixed point. This fixed point is stable, as every initial condition will come to the satet of pure rolling.

According to the plot generated by code, we can see both the angular velocity and linear velocity stay constant at the same point(see Fig.3(a)).

Analytically, we see once $v + w \cdot r$ reached zero, which means the contact point has no velocity with respect to the table.(The physical terminology is pure rolling[1]). At this point, both $\frac{dv}{dt}$ and $\frac{dw}{dt}$ vanishes **i.e.** the linear velocity and angular velocity will not change anymore. One extreme case is $v_0 + w_0 \cdot r = 0$, it will reach the state of pure rolling when $t = 0$ (see Fig.3(b))

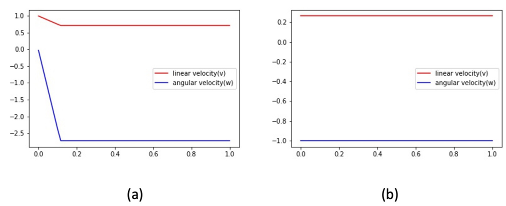

Figure 3: Plots of v and ω

Nevertheless, it is not possible that the ball will keep rolling without stopping. In the next section, we'll investigate how the ball stop.

Rolling resistance

In last section, we have shown that if we only consider the sliding friction, the motion of the ball will keep constant once $v + w \cdot r$ vanishes **i.e.** the ball will keep rolling at that speed. If both table and the cue ball are ideal rigid bodies, then it is the case. However, it is not possible in the real world, as no material is ideally rigid. In particular, the cloth of the table is far from rigid. As is shown in fig [], a small part of the ball is contained in the cloth, which triggers resistance other than sliding friction. According to [], as this force is comparable to W_{object}, we can analogously define its effect on object by defining its friction coefficient. Namely, we assume the rolling friction is $f_r = \mu_r \cdot mg$.

Here are two things we need to clarify.

- As $\mu_{rolling} \ll \mu_{sliding}$ (usually $\mu_{rolling}$ has scale of 10^{-4} to 10^{-2}, while $\mu_{friction}$ has scale of 10^{-1}), we don't take rolling friction into consideration before pure rolling.
- Rolling friction reduce v and w simultaneously. Therefore, we assume the ball is always pure rolling.

Hence, the motion of cue ball after pure rolling is governed by:

$$\frac{dv}{dt} = -\mu_r g$$

$$\frac{d\omega}{dt} = \frac{\mu_r g}{r}$$

Motion of the cue ball

Therefore, we can combine the conclusions of above two parts. The motion of the cue ball is governed by:

If $sgn_f(v,\omega) = -sgn(v + w \cdot r) \neq 0$,

$$\frac{dv}{dt} = sgn_f(v,\omega) \cdot \mu g$$

$$\frac{d\omega}{dt} = sgn_f(v,w) \cdot \frac{5\mu g}{2r}$$

If $sgn_f(v,\omega) = -sgn(v + w \cdot r) = 0$,

$$\frac{dv}{dt} = -\mu_r g$$

$$\frac{d\omega}{dt} = \frac{\mu_r g}{r}$$

For now, we don't know the exact value of the constant parameters μ, μ_r, and r.

Cueing process

Here we come to the first part. Given the impulse of J and the position of hitting, what kind of initial velocity and initial angular velocty will it trigger? (Un)Fortunately, this problem is perfectly done by early works []. The only effort I did is to generalize the table he gives to one equation and make it consistent with the notation that I use. After transforming, v_0 and ω_0 is determined by:

$$v_0 = \frac{J}{m}$$
$$\omega_0 = -\frac{5hJ}{2mr^2}$$

where J is the impulse the cue stick provide, and the h denotes the hitting point(with respect to the middle point of the cue ball), $h \in [-r, r]$.

Estimation of parameters

For now, we have solve the motion of the ball analytically. However, to obtain a full picture of the motion of cue ball or to make usage of it(**i.e.** predict the ball's motion and position at any given time), we need to know the value/range of parameters.

Here we take **snooker** as an example.

Parameters: m, r, J, μ, μ_r.

Estimating m, r, and J

The mass m and r can be obtained directly from internet [3].

$$m = 0.1545(kg), r = 0.2625(m)$$

Though it is not necessary to know J as it is an input, it is good to know the upper-bound of J.(**e.g.** we can know how long can a cue ball go at maximum). As the fastest velocity of a cue ball recorded is $16(m/s)$, therefore, the maximum of the impulse

$$J_{max} = m \cdot v_{max} = 2.472kg \cdot m/s$$

Estimating μ

One of the most interesting part. Generally speaking, the friction coefficient is determined by the material of two objects, but the formula for it is essentially missing. One way to do it is to do conducting by direct experiment: dragging a ball on the table by a spring-loaded meter and read the force when the ball has uniform linear motion. However, this method turns out to be impractical. On the one hand, it is hard to make the cue ball sliding on the table with out sliding. On the other hand, it is hard to determine if the velocity is uniform.

In this project, we propose a method to estimates μ based on the motion of the cue ball. The idea is quite easy. If we can get the acceleration rate of a ball.

In the remaining part of this section, we propose a method to find the sliding friction coefficient μ.

- Find a video clip of a long-distance backward spinning shot (it means make the cue ball hit the target ball in the long distance and the cue ball goes back).
- By restrict it to backward spinning, we make sure it is decelerating linearly before it hits the ball. The backward spinning indicates the cue ball has backward spinning $\omega > 0$ before hitting the target, which means it has initial angular velocity $w_0 > 0$ and $w > 0$ along the way. It follows that $sgn_f = -sgn(v + wr) = -1$ all the time **i.e.** $\frac{dv}{dt} = -\mu g, \forall t$. Thus, the cue ball is decelerating linearly. The reason we need a long-distance is the change of velocity is more obvious, which reduces the error.
- Extract all frames of that video clip.
- choose two consecutive frame of after the cueing, and two consecutive frame before hitting the target ball.
- Stacking the four images together and we are able to see the position of the cue ball at those 4 frames. Denote the pixel coordinates of four positions $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$.

- Denote the duration of $\mathbf{x}_1$ and $\mathbf{x}_2$ (or $\mathbf{x}_3$ and $\mathbf{x}_4$) Δt. The average velocity between $\mathbf{x}_1$ and $\mathbf{x}_2$ can be obtained by

$$v_a = \frac{||\mathbf{x}_1 - \mathbf{x}_2||_2}{\Delta t}$$

$$v_b = \frac{||\mathbf{x}_3 - \mathbf{x}_4||_2}{\Delta t}$$

  where $||\cdot||_2$ is the euclidean distance(L_2 norm). **Note that** here we need to convert the pixel length to actual length. It is accessible, as we know both the pixel length and actual length of the width of the snooker table.

- These two average velocity can be regarded as the velocity of mid point of $\mathbf{x}_1$ and $\mathbf{x}_2$ (or $\mathbf{x}_3$ and $\mathbf{x}_4$), and the distance of the two points can be obtained by

$$d = ||\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \frac{\mathbf{x}_3 + \mathbf{x}_4}{2}||_2$$

  Similarly, we need to convert it to the actual length.

- μ can be derived by **Kinetic Energy Theorem**,

$$\frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2 = -m\mu g d$$

$$\mu = -\frac{v_a^2 - v_b^2}{2gd}$$

Though the rigorous process is lengthy, the punchline is quite clear. By finding a clip of long-distance backward spinning, we can find a period when the cue ball is linearly decelerating, where μ is easy to solve.

Figure 4: Extracted Frames

Here we use the video clip of the game between Ding Junhui and Ronnie O'Sullivan at world snooker championship 2021[]. We extract frame 166, 168, 186 and 188 (see Fig.4).

For easy visualization, we stack 4 images together and label the pixel coordinates (see Fig.5). By computation described in the above processes, we get

Figure 5: Stacked images (with pixel coordinates)

the value of sliding friction coefficient, which is

$$\mu = 0.2520$$

Estimating μ_r

The estimation of rolling friction is quite cumbersome. [2] proposed a method of calculating μ_r from μ based on geometric relations. However, this method is not applicable here it is not possible to know the proportion of the cue ball immersed into the cloth.

Therefore, we are only able to estimates μ_r like how we estimates μ above. The problem is, we are not able to detect when does the cue ball starts to do pure rolling. In that case, we find a clip which the ball's velocity is small but rolls really long, to make sure the ball is pure rolling when it is about to stop.

The video clip we use is from the game between O'Sullivan and Selby at Final of 2020 Scottish Open [5]. We omit the process of this computation, which is identical to the process of measuring μ. The result we get is

$$\mu_r = 0.0184$$

The value obtained is consistent with our assumption that $\mu_r \ll \mu$.

Numerical, Implications, and Visualization

Numerical

Equipped with the analytical relations of v and ω and estimations of all constant parameter, we are able to do the numerically simulate the model. As the growth are basically constants (though with some discontinuities), we didn't encounter problems when using numerical methods. The details of the numerical simulations is in the jupyter notebook script `velocity.ipynb`. The basic idea is simulate the part before and after the pure rolling state seperately by the package `scipy.integrate.odeint`.

Here's an example (see Fig.6, here we set $J = 0.5kg \cdot m/s$, $h = -0.2m$, which is a typical backward spining)

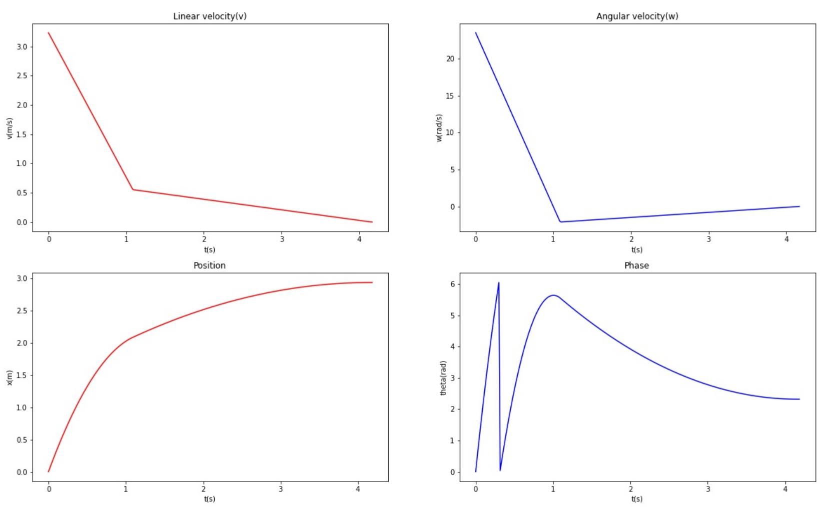

Figure 6: Numerical Estimation of the model ($J = 0.5kg \cdot m/s$, $h = -0.2m$)

From the numerical stimulation, one is able to answer many questions, like when does the ball starts pure rolling, when does it stop. However, these questions are also solvable from analytical solutions(see [1][2]). For more examples of simulation, see the folder `numerical`.

Visualization: Generating a video clip

Here comes the most interesting part of the project. In this part, we want to design a program that generates the video depicting cue ball's rotation **i.e.** how the cue ball moving and rotating.

The moving side is easy to tackle, while visualizing rotating can be troublesome. One way to do it is to mark a point on the circle. Its trajectory with respect to the table when cue ball is moving is analytically solvable and can be numerically approximated. We can thus putting the movement of the point and the movement of the ball together, which will trigger the sense of rotation vision.

However, we use a more natural method. Given the v and ω at any time, we are able to predict the position and the phase of the ball at any time. Therefore, we can generate the picture of the ball at any single frame. Also, we can express the rotation by drawing an arrow on the ball. By concatenating those frames together, we get the video.

The file for generating the clip is `video_generator.py` in the folder `video-generator`, together with a demo.

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References

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